Lecture 11 Angular Momentum.

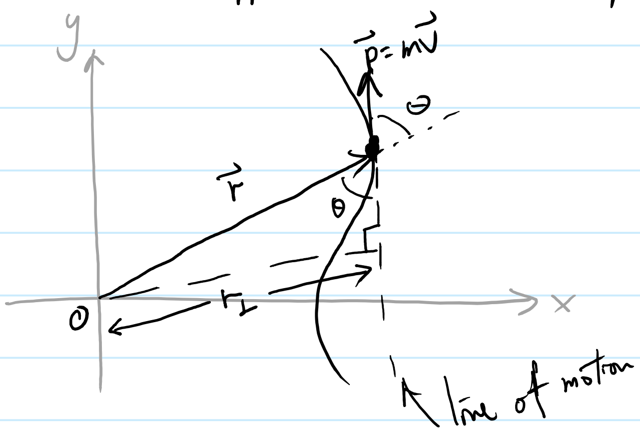
So far most of the linear kinematic and dynamic quantities have their counterparts in rotational motion.

Similar to $\vec{F} \longrightarrow \vec{\tau} = \vec{r} \times \vec{F}$

define: $\vec{p} \longrightarrow \boxed{\vec{L} = \vec{r} \times \vec{p}}$

angular momentum of a particle.

Suppose $\vec{r}$ and $\vec{p}$ are on x-y plane.



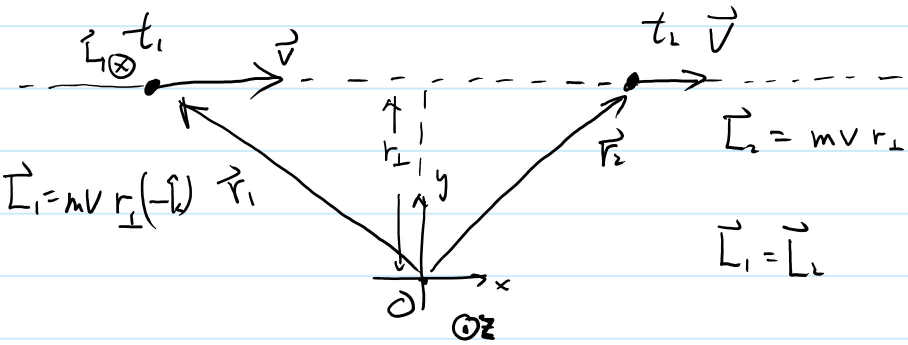
$\vec{L} = \vec{r} \times \vec{p}$

$L = r \cdot p \cdot \sin\theta = r_\perp \cdot p = mvr_\perp$

Properties:

- $\vec{L}$ depends on where O is.
- particle needs not to be rotating about O in order to have non-zero $\vec{L}$.
- particle moving in a straight line has constant $\vec{L}$!

Consider.

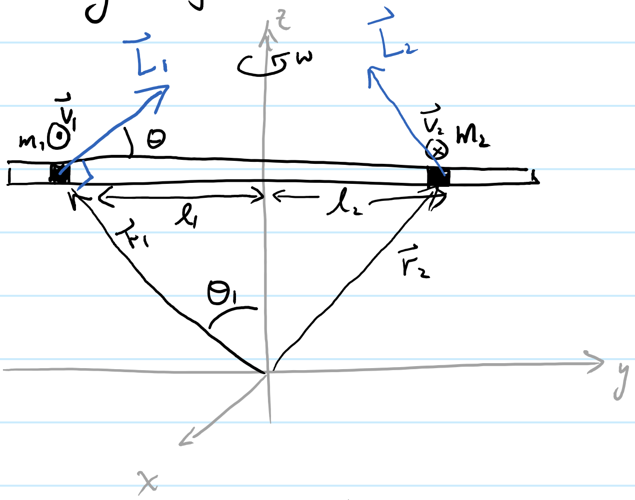


$\vec{L}_1 = mv\,r_\perp(-\hat{k})$

$\vec{L}_2 = mv\,r_\perp(-\hat{k})$

$\vec{L}_1 = \vec{L}_2$

Rigid Body (fix axis)

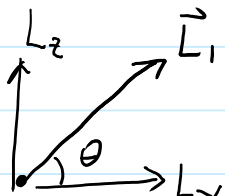


$\vec{L}_1 = \vec{r}_1 \times m_1 \vec{V}_1$

$L_1 = r_1 m_1 v_1$ $\qquad v_1 = l_1 \omega$

$= m_1 r_1 (l_1 \omega)$

$= m_1 l_1 r_1 \omega$



z-component of $\vec{L}_1$: $\quad L_{1z} = L_1 \sin\theta$

$= m_1 l_1 \underbrace{r_1 \omega \sin\theta}_{l_1}$

$L_{1z} = m_1 l_1^2 \omega .$

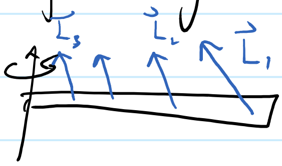
Z-component of $\vec{L}_{tot} = \sum_i \vec{L}_i$: $\quad L_z = \sum_i m_i l_i^2 \omega$

$$\boxed{L_z = I_z \omega}$$

y-component of $\vec{L}_{tot}$ will be cancelled <u>if</u> the rod is symmetrical about the rotation axis.

$\Rightarrow$ only L_z is non-zero.

Example of asymmetric rotation.



$L_y = \sum L_{iy} \neq 0$

but $\underline{L_z = I_z \omega}$ <u>still</u>

Relation to torque   recall $\vec{F} = \frac{d\vec{p}}{dt}$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\Rightarrow \quad \frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{v} \times m\vec{v} + \vec{r} \times \vec{F}_{net}$$

$(\vec{v} \times \vec{v} = 0)$

$$= 0 + \vec{\tau}_{net}$$

$$\Rightarrow \quad \boxed{\vec{\tau}_{net} = \frac{d\vec{L}}{dt}}$$

for Rigid Body:

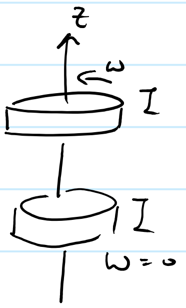
$$\vec{\tau}_{net} = \sum \vec{\tau}_{i\,net} = \underbrace{\sum \vec{\tau}_{ext}}_{\text{only external torque}} = \frac{d\vec{L}_{tot}}{dt}$$

$\vec{L}$ of the system.

If $\sum \vec{\tau}_{ext} = \vec{0} \Rightarrow \underline{L_{tot} = \text{constant}}$

conservation of angular momentum.

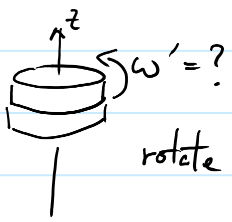
Examples



initial

suppose $\vec{\tau}_{ext} = \vec{0} \Rightarrow \Delta\vec{L} = \vec{0}$

$$L_{i_z} = I\omega + I \cdot 0 = I\omega$$



final

$$L_{f_z} = I_f \omega' = (2I)\omega' = 2I\omega'$$

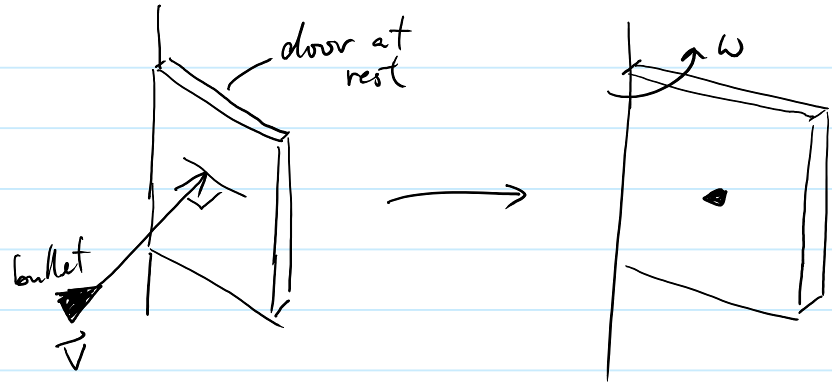
rotate together $\Rightarrow$

$$L_{i_z} = L_{f_z}$$

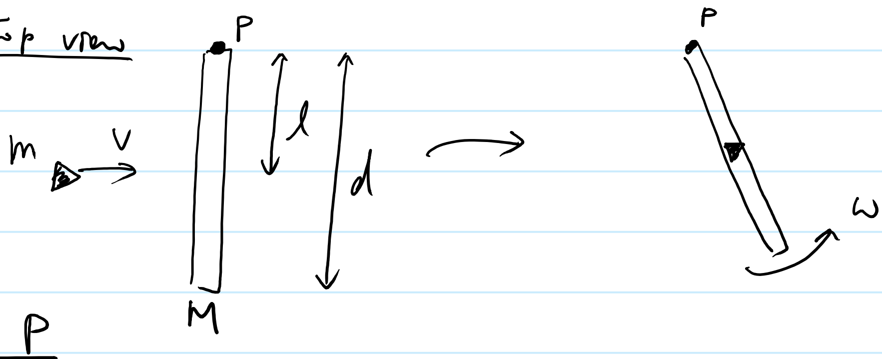
$$I\omega = 2I\omega'$$

$$\omega' = \frac{\omega}{2}$$

"completely inelastic collision" in rotation



<u>top view</u>



<u>about P</u>

<u>Before collision</u>

$$L_i = mlv + I_{door}\cancel{\omega_0}^{\to 0} \qquad \text{where} \quad I_{door} = \frac{1}{3}Md^2$$

$$L_i = mlv$$

<u>After collision</u>

$$L_f = I_{tot}\,\omega = \left(ml^2 + I_{door}\right)\omega$$

$$= \left(ml^2 + \frac{1}{3}Md^2\right)\omega$$

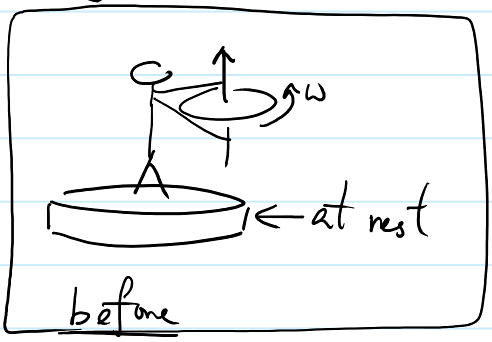
Since $\tau_{ext} = 0$ (smooth hinge at P)

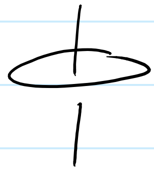
$$\Rightarrow \quad L_i = L_f$$

$$\Rightarrow \quad mlv = \left(ml^2 + \frac{1}{3}Md^2\right)\omega$$

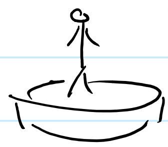
$$\omega = \frac{mlv}{ml^2 + \frac{1}{3}Md^2}$$

Standing on spinning top



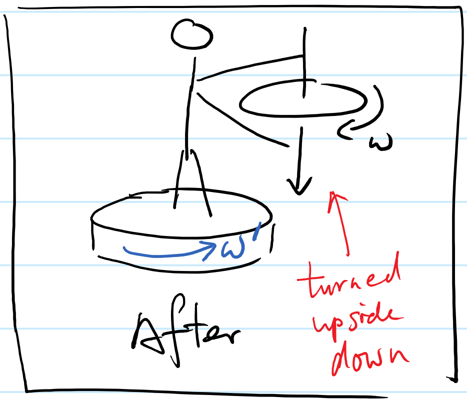


I_{wheel}



I_{other}.

(take upward as positive)



$$L_i = I_{wheel} \cdot \omega + I_{other} \cdot 0$$

$$L_f = -I_{wheel}\,\omega + I_{other}\,\omega'$$

$$L_i = L_f$$

$$\Rightarrow \quad I_{wheel}\,\omega = -I_{wheel}\,\omega + I_{other}\,\omega'$$

$$\Rightarrow \quad \omega' = \frac{+2 I_{wheel}\,\omega}{I_{other}}.$$

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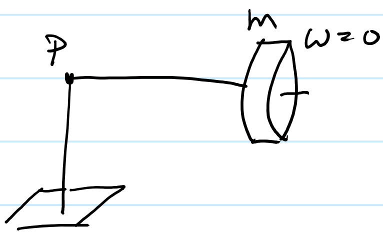
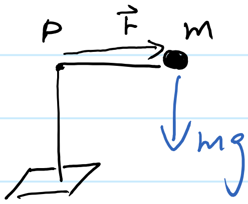
when $\vec{\tau} \neq \vec{0} \quad \Rightarrow \quad \Delta\vec{L} \neq \vec{0}$

$$\Delta\vec{L} = \int_t^{t+\Delta t} \vec{\tau} \cdot dt \approx \vec{\tau}\,\Delta t \qquad\qquad \vec{\tau} = \frac{d\vec{L}}{dt}$$

Case 1



$=$

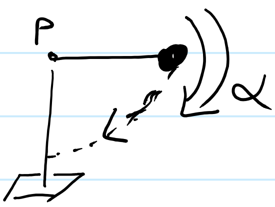


$\otimes \vec{\tau}$

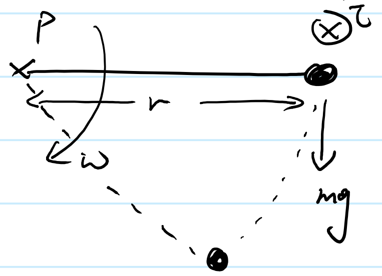
$\vec{\tau}$ on the wheel due to gravity : $\vec{\tau} = \vec{r} \times m\vec{g}$

into the page $\otimes$

$\vec{\tau} = I\vec{\alpha} \quad \Rightarrow$



If we consider the mass and the rod (massless) holding it as a rigid body.



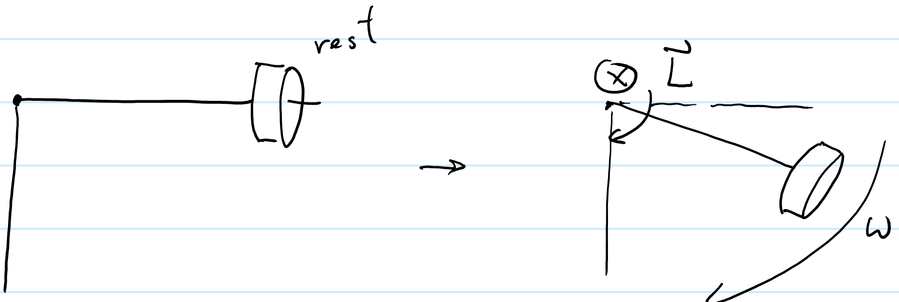
$$I_p = m r^2$$

$$L = I_p \omega$$

angular velocity of the (rod & mass)

$$\vec{\tau} = \frac{d\vec{L}}{dt} = I_p \frac{d\vec{\omega}}{dt} = I_p \vec{\alpha} \quad \text{same as before.}$$

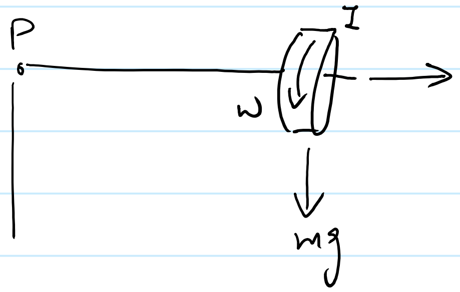
But we can view the swing down motion as an increase of angular momentum towards the page due to $\vec{\tau}$.



$L = 0$ $\qquad\qquad$ $L = I_p \omega$

$\vec{\tau}$ makes $\vec{L}$ increase towards the page. $\otimes$

Precession of gyroscope.



$\vec{L} = I\omega$, I = moment of inertia of the wheel.

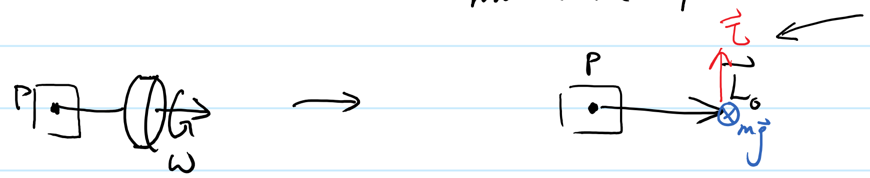
The weight introduces a torque to the system into the page ⊗

Since the system has an initial angular momentum $\vec{L}_0$ as the wheel is rotating about its axis and the torque is perpendicular to its $\vec{L}_0$, the torque does not increase the magnitude of $\vec{L}$ but changes the direction of $\vec{L}$.

top view

angular momentum space

$\Delta\vec{L} = \vec{\tau}\,\Delta t$

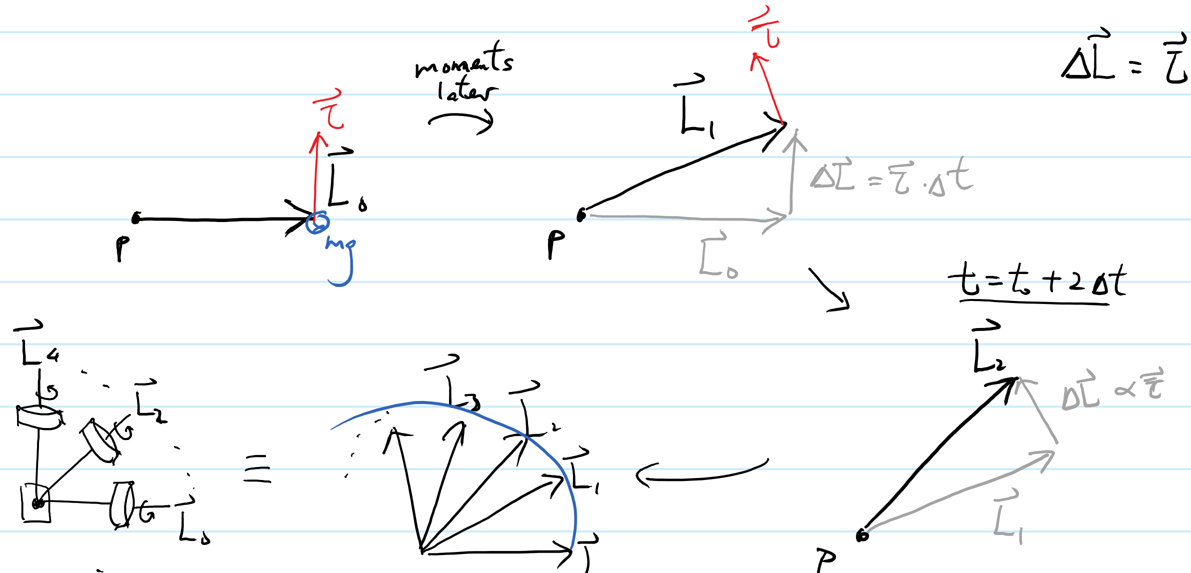


$t = t_0$

$t = t_0 + \Delta t$

change in $\vec{L}$

$\Delta\vec{L} = \vec{\tau}\,\Delta t \;//\; \vec{\tau}$



precession